Asymptotic Optimality of AO-x

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Asymptotically optimal x (AO-x) [1] is a meta kinodynamic motion planner [4], which can take as input any feasible kinodynamic motion planner, and convert it into an asymptotically optimal planner. Common choices for x are the rapidly-exploring random tree (RRT) [3] planner or the expansive space-trees (EST) [2] planner. In this document, I will summarize the proof of asymptotic optimality for AO-x, which is fully detailed in [1].

Let us first declare some variables. Let X be a feasible kinodynamic motion planning algorithm. Note that X needs to be a probabilistically complete kinodynamic planner (but not necessarily asymptotically optimal).

To prove that AO-x with algorithm X is asymptotically optimal, let us first write down two assumptions:

Assumption 1. *Algorithm X will terminate in finite time.*

Assumption 2. *Algorithm X reduces cost by a non-negligible amount. This means: Let $\bar{C}$ be the cost limit, and $C^\star$ the optimal cost. Then the expected suboptimality is shrunk toward $C^\star$ by a non-negligible amount each iteration. If $C(\pi)$ is the current path cost, then*

$$\mathbb{E}[C(\pi) \mid \bar{C} - C^\star] \leq (1 - w)(\bar{C} - C^\star), \tag{1}$$

whereby $w > 0$ is a constant positive value.

By assuming Assumption 1 and 2, we can state and prove the following theorem.

Theorem 1. *AO-x is asymptotically optimal.*

Proof. Let $S_0, \ldots, S_n$ be non-negative random variables denoting $C(\pi_i) - C^\star$, i.e. the cost difference between the cost of the returned path in iteration i and the optimal path cost $C^\star$.

Our goal is to prove that the sequence converges almost surely, i.e.

$$P(\lim_{n\to\infty} S_n = 0) = 1. \tag{2}$$

This means, that the sequence $S_0, \ldots, S_n$ will converge to 0 when n goes to infinity with probability 1. The proof itself consists of 4 steps.

Step 1: Transformation to convergence in probability

Let us first make a transformation of Eq. (2). Since S_n is always nonnegative, it is sufficient to show that S_n converges in probability (see info box below) as $S_n \to 0$, to prove the original result which results in

$$\lim_{n\to\infty} P(S_n > \epsilon) = 0. \tag{3}$$

While this is a weaker statement (in probability) compared to Eq. (2) (almost surely), we can, however, rely on the fact that if a nonnegative sequence (like S_n) converges to 0 in probability, then there exists a subsequence that actually converges to 0 with probability 1 [5]. From here, we are left to prove that the sequence converges in probability.

> **Convergence in Probability.** Let $X_1, X_2, \ldots$ be a sequence of random variables. We say this sequence *converges in probability* to a random variable X, written as $X_n \xrightarrow{p} X$, if, for all $\epsilon > 0$
>
> $$\lim_{n\to\infty} P(|X_n - X| > \epsilon) = 0.$$

Step 2: Applying Markov inequality

The Markov inequality states that $P(S_n \geq \epsilon) \leq \frac{\mathbb{E}[S_n]}{\epsilon}$. If we could prove that

$$\lim_{n\to\infty} \frac{\mathbb{E}[S_n]}{\epsilon} = 0, \tag{4}$$

then this would imply Eq. 3. This is a valuable step, since our assumptions already provide an upper bound for the expected value.

Step 3: Find upper bound to conditional expectancy

By using Assumption (2), we can find an upper bound to the expected value of S_n as

$$\begin{aligned}\mathbb{E}[S_n] &= \int \mathbb{E}[S_n \mid S_{n-1}] P(S_{n-1}) dS_{n-1} && (5)\\ &\leq (1-w) \int S_{n-1} P(S_{n-1}) dS_{n-1} && (6)\\ &= (1-w)\, \mathbb{E}[S_{n-1}] = (1-w)^n\, \mathbb{E}[S_0] && (7)\end{aligned}$$

Step 4: Evaluate the expectancy

Using the upper bound for the expectancy we can now evaluate

$$\begin{aligned}\lim_{n\to\infty} P(S_n > \epsilon) &\leq \lim_{n\to\infty} \frac{\mathbb{E}[S_n]}{\epsilon} && (8)\\ &\leq \lim_{n\to\infty} \frac{\mathbb{E}[S_0](1-w)^n}{\epsilon} && (9)\\ &\leq \frac{\mathbb{E}[S_0]}{\epsilon} \lim_{n\to\infty} (1-w)^n = 0. && (10)\end{aligned}$$

This in turn verifies Eq. 2 through Eq. 3 as desired. □

References

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