Geometry of quaders on top of surface elements of polytopes

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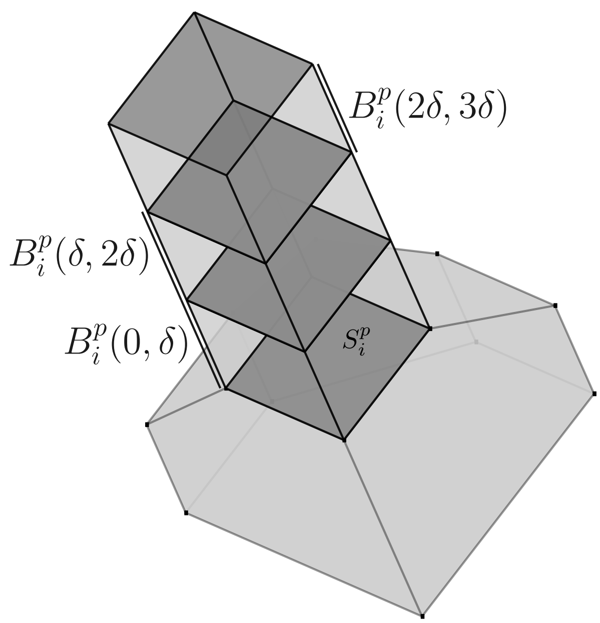

Let us give an explicit expression for a quader which is hovering on top of a surface element of a polytope. First we start with the definition of a polytope [Boyd and Vandenberghe(2004)]:

Definition 1. *Let an object O_i be a convex bounded polytope*

$$O_i = \{x \in \mathbb{R}^3 | a_j^{(i)T} x \leq b_j^{(i)}, \|a_j^{(i)}\|_2 = 1, j \in [1, M_i]\} \tag{1}$$

with M_i the number of halfspaces.

Let us take one surface element S_i^p of O_i. This is defined as:

Definition 2 (Surface Element). *Given an object O_i, we call*

$$\begin{aligned} S_i^p = \{x \in \mathbb{R}^3 | a_p^{(i)T} x = b_p^{(i)}, a_j^{(i)T} x \leq b_j^{(i)}, \\ j = 1, \cdots, p-1, p+1, \cdots, M_i\} \end{aligned} \tag{2}$$

the p-th surface element of object O_i, and $a_p^{(i)}$ is the surface normal with distance $b_p^{(i)}$ to the origin.

A quader on top of S_i^p can now be defined as

Definition 3. *The quader $B_i^p(\Delta_L, \Delta_U)$ of height $\delta = \Delta_U - \Delta_L$ located with distance Δ_L above S_i^p is defined as the set of points in*

$$
\begin{aligned}
B_i^p(\Delta_L, \Delta_U) = \{x \in \mathbb{R}^3 | & -a_p^{(i)T} x \leq -b_p^{(i)} - \Delta_L, \\
& a_p^{(i)T} x \leq b_p^{(i)} + \Delta_U, \\
& \hat{a}_j^{(i)T} x \leq \hat{b}_j^{(i)}, \\
& {}_{j=1,\cdots,p-1,p+1,\cdots,M_i}\}
\end{aligned} \tag{3}
$$

with $\hat{a}_j^{(i)}, \hat{b}_j^{(i)}$ belonging to the projected hyperplane j, with

$$
\begin{aligned}
\hat{a}_j^{(i)} &= a_j^{(i)} - (a_j^{(i)T} a_p^{(i)}) a_p^{(i)} \\
\hat{b}_j^{(i)} &= \hat{a}_j^{(i)T} x_{j,0}^{(i)}
\end{aligned} \tag{4}
$$

whereby $x_{j,0}^{(i)}$ is one point on the intersection between hyperplane H_j and surface element S_i^p

$$
\begin{aligned}
x_{j,0}^{(i)} \in \{x \in \mathbb{R}^3 | & a_j^{(i)T} x = b_j^{(i)}, \\
& a_k^{(i)T} y \leq b_k^{(i)}, \\
& {}_{k=1,\cdots,j-1,j+1,\cdots,p-1,p+1,\cdots,M_i}\}, \\
& a_p^{(i)T} y = b_p^{(i)}, \\
& \|x - y\|^2 = 0\}
\end{aligned} \tag{5}
$$

Note that x_0 does only exist, when there is a common border between S_i^p and H_j. If there is no border, then $\hat{a}_j^{(i)}$ and $\hat{b}_j^{(i)}$ do not exist, i.e. they are not halfspace intersections of the box.

References

[Boyd and Vandenberghe(2004)] Stephen Boyd and Lieven Vandenberghe. *Convex Optimization*. Cambridge university press, 2004.