Introduction to Configuration Spaces and Homeomorphisms

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One of the fundamental concepts in robotics are configuration spaces [3]. This document gives a gentle introduction to them including a discussion about the topological equivalence of spaces. I assume a basic mathematics background on the level of a first-year university student.

1 Configuration Spaces

A configuration space represents the set of all possible configurations a robot can achieve. A configuration thereby defines the position and orientation of the robot's entire mechanical structure by specifying the values of all its degrees of freedom (DoFs). A DoF is an independent parameter governing a robot's movement or actuation, such as a drawer's linear motion or a motor's rotation. Collectively, these DoFs describe the robot's geometry, enabling the configuration space to capture every possible way the robot can be positioned in its environment.

Let us start with a simple example. Consider a lamp that is stationary, and which has a single switch which can be either `on` or `off`. The set of all configurations of this lamp is therefore the set $\{\texttt{on}, \texttt{off}\}$. The configuration space of a lamp is therefore

$$X = \{\texttt{on}, \texttt{off}\}. \tag{1}$$

Let us increase the complexity of this example. Instead of a discrete switch, let us assume that the lamp has a continuous dimmer switch, which can be turned from 0 to 1. The corresponding configuration space is then the interval $[0, 1]$ as

$$X = [0, 1]. \tag{2}$$

Often robots have more than one DoF, like a disk on a planar floor. Such a disk can move into two directions. If we denote each floor direction by one DoF in $[0, 1]$, we can describe the configuration space of this disk using the Cartesian product to get

$$X = [0, 1] \times [0, 1]. \tag{3}$$

It is useful to think about the Cartesian product as a physical process whereby we go over each element in $[0, 1]$ and attach to each of them a copy of the interval $[0, 1]$.

Let us take the disk one step further by transforming it into a car on the floor. This car can be oriented into different directions using an orientation parameter $\theta \in [0, 2\pi)$, whereby $[0, 2\pi)$ is the set of all orientations whereby 0 and 2π yield the same value. The configuration space is the Cartesian product of all parameters as

$$X = [0, 1] \times [0, 1] \times [0, 2\pi). \tag{4}$$

Let us make this example even more complex, such that we get closer to a realistic robot platform. Imagine that the car is instead a swivel chair. This chair can additionally rotate around its platform and might have an adjustable back, which can be moved forward and backward. Those add two additional DoF to the configuration space to give us

$$X = [0, 1] \times [0, 1] \times [0, 2\pi) \times [0, 2\pi) \times [0, 1]. \tag{5}$$

Note that this chair requires five parameters (DoF) to uniquely define all possible configurations. We say that the configuration space is of dimension five.

Question 1 *Can you define the configuration space of a car with two doors and a trunk lid? How many total DoF does this car have?*

Question 2 *Can you define the configuration space of a spherical balloon floating through 3-dimensional space? Can you do the same for a Zeppelin? Why do those two configuration spaces differ?*

Given that we now have a basic intuition about configuration spaces, we can define them mathematically as

Definition 1 (Configuration Space) *A configuration space X of dimension n consists of variables $x = (x_1, \ldots, x_n)$ with domains $x_i \in X_i$, whereby each X_i represents a degree of freedom.*

2 Topological Equivalence of Configuration Spaces

It turns out that it is very useful to classify configuration spaces based on their structure. This can be important for different reasons:

- If two configuration spaces are equivalent, we might be able to transfer an algorithm from one space to the other—instead of creating a new algorithm.
- If the configuration spaces are equivalent, we can often show that the topological complexity of algorithms on those spaces is similar [1, 2].

- It allows us to quickly see the difference or commonality between two problems, while removing the unnecessary details of the concrete configuration space involved.

What kind of classification do we want to have? Here are some requirements.

- **Scale-Invariant** If we have two 1-DoF robots (think about a point on a line) with different intervals $[a, b]$ and $[c, d]$, then those are very similar problems which only differ by a scaling from the real number line $\mathbb{R}$. This suggests that spaces should be equivalent under scaling.

- **Bending** Imagine two 1-Dof robots, one moving on a circle, one moving on an ellipse. Those problems look the same when bending the spaces into each other. We would want such spaces to also be equivalent.

- **Path-Connectivity** If two configuratons are connected by a path, then those configurations should remain path connected when transformed. Thus we want to prevent cutting or gluing of spaces, which would remove or add path-connectivity.

To adhere to those requirements, we need an equivalence relation on configuration spaces, which retains path-connectivity while allowing for scaling and bending. This is where homeomorphisms come in. A homeomorphism is a bijective, continuous transformation with continuous inverse that preserves topological properties like path-connectivity [4]. It turns out that homeomorphisms not only adhere to the requirements listed above, but that homeomorphisms are also act as equivalence classes of spaces, meaning they create a partitioning of spaces.

If a homeomorphism exists between two spaces, we call them homeomorphic. This is basically the topological way of saying that they are equivalent. Let X and Y be two configuration spaces. We say that they are homeomorphic, if there exists a mapping H from X to Y with the following properties:

- **Bijective** It is one-to-one (injective) and onto (surjective). To prove that it is injective, you can show that if $f(x) = f(y)$ then it must follow $x = y$. For being surjective, show that if $y \in Y$ is an arbitrary element of Y, then there exists an $x \in X$ such that $f(x) = y$.

- **Continuous**. Prove that $f(x)$ is continuous. There are many ways to do that, including topological arguments [4], using the metric space definition, or showing that a function is entirely composed of continuous functions.

- **Continuous Inverse**. Find the inverse $f^{-1}(x)$ and prove that this function is continuous, too.

Using homeomorphisms, we can relate the configuration spaces we found earlier to a small set of prototype spaces.

3 Prototype Spaces

To classify spaces based upon homeomorphisms, it is useful to find transformations to some of the prototype spaces in mathematics. This includes spaces like the n-dimensional real number line $\mathbb{R}^n$, the n-dimensional sphere $\mathbb{S}^n$, the n-dimensional torus $\mathbb{T}^n$, and the n-dimensional space of binary elements $\mathbb{Z}_2^n$. Note that the definition of those spaces follows the same logic which is $\{\text{Symbolic name}\}^{\{\text{dimensions}\}}$.

Let us go over the spaces above and relate them to one of those prototype spaces. The single-switch lamp has two possible states, therefore two elements. It is thus equivalent to

$$X = \mathbb{Z}_2 \tag{6}$$

This equivalence can be seen by creating a simple homeomorphism $H(\texttt{off}) = 0$ and $H(\texttt{on}) = 1$. It turns out that this is continuous since the space itself only consists of those two elements.

When using the continuous dimmer switch, we defined the configuration space as the interval $[0, 1]$. It can be shown that $[0, 1]$ can continuously be transformed into the (extended) real number line $\mathbb{R}^1$[1]. To show this, we need to define an explicit mapping. One possible choice is

$$H(x) = \frac{1}{2}\left(\frac{x}{1 + |x|} + 1\right). \tag{7}$$

Question 3 *Show that H is indeed a homeomorphism.*

Thus we can write $X = \mathbb{R}^1$. A similar argument can be used for the disk on the planar floor, where $X = [0, 1] \times [0, 1]$ is equivalent to $X = \mathbb{R}^2$.

The next configuration space is the car $X = [0, 1] \times [0, 1] \times [0, 2\pi)$. This involves $[0, 2\pi)$, the set of all orientations in 2D. Since 0 and 2π yield the same value, they can be identified [3]. It turns out that this is equivalent to the space $\mathbb{S}^1$, the circle. Thus we can write

$$X = \mathbb{R}^2 \times \mathbb{S}^1. \tag{8}$$

Question 4 *Which prototype spaces is the configuration space of the swivel chair equivalent to?*

Question 5 *Is the circle $\mathbb{S}^1$ and the interval $[0, 1]$ equivalent? Why?*

References

[1] Michael Farber. Topological complexity of motion planning. *Discrete & Computational Geometry*, 29(2):211–221, 2003.

[1] Note that we define the real number line here as the extended number line which is $\mathbb{R} \cup \{-\infty, +\infty\}$. This ensures that the number line is compact and can be related to compact sets [4]

[2] Michael Farber. *Invitation to Topological Robotics*. EMS Press, Zürich, 2008.

[3] Steven M LaValle. *Planning algorithms*. Cambridge university press, 2006.

[4] James Raymond Munkres. *Topology*. Prentice Hall, Upper Saddle River, NJ, 2 edition, 2000.