Maximum Ball on a Surface Element of a Polytope

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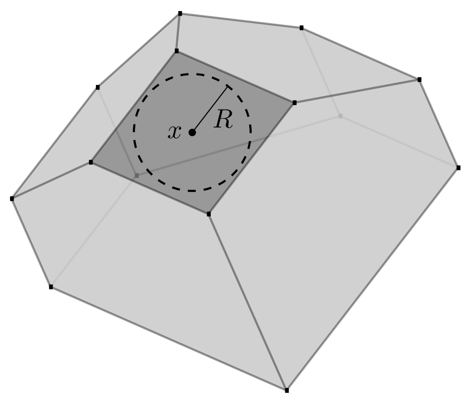
September 2014



Let us start with considering the problem of finding the maximum ball $\mathcal{B}_R = \{x \in \mathbb{R}^N | \|x\| \leq R\}$, inside a polytope $\mathcal{P} = \{x \in \mathbb{R}^N | a_i^T x \leq b_i, i = 1, \cdots, M\}$. As explained in [Boyd and Vandenberghe(2004)], this problem can be formulated as a linear programming problem. The main insight is: Each hyperplane of the polytope can be seen as a tangential plane for a specific ball radius. The shortest distance between the tangential plane and the center of the ball is given by the radius multiplied by the normal of the plane. To see if the ball is inside of the halfspace defined by the hyperplane, we need to check the shortest distance only. Let us denote this intersection of ball with the hyperplane normal as $v_i = y + R\frac{a_i}{\|a_i\|}$, with y being the center of the ball. Intuitively, the ball is inside of the polytope if and only if (iff) each v_i is inside of the polytope, meaning we need to have the inequalities $a_i^T v_i \leq b_i$ fulfilled. This gives us the linear programming formulation which can be found in [Boyd and Vandenberghe(2004)]:

$$\begin{aligned} \underset{y \in \mathbb{R}^N, R \in \mathbb{R}}{\text{maximize}} \quad & R \\ \text{subject to} \quad & a_i^T y + R\|a_i\|_2 \leq b_i \\ & R \geq 0 \end{aligned} \tag{1}$$

Let us now consider the problem of finding the maximum ball in dimension $N-1$, $\mathcal{B}_R = \{x \in \mathbb{R}^{N-1} | \|x\| \leq R\}$, restricted to a surface element of $\mathcal{P}$, which we denote as

$$\mathcal{S} = \{x \in \mathbb{R}^N | a_p^T x = b_p, a_i^T x \leq b_i, i = 1, \cdots, p-1, p+1, \cdots, M\}$$

Meaning, the surface element $a_p^T x = b_p$ will be intersected by the other hyperplanes, which essentially makes this problem a lowerdimensional one. Since we are operating on our surface element, we are interested in finding the normal vectors, lying on the surface and pointing towards the intersection of the other hyperplanes. Meaning, instead of having $v_i = y + R\frac{a_i}{\|a_i\|}$, we need to find the orthogonal projection of a_i onto the surface element, which we get by $v_i = y + R\frac{a'_i}{\|a'_i\|}$ with $a'_i = a_i - (a_i^T a_p)a_p$ being the orthogonal projected vector. The resulting linear optimization problem is

$$\begin{aligned} \underset{y \in \mathbb{R}^n, r \in \mathbb{R}}{\text{maximize}} \quad & R \\ \text{subject to} \quad & a_i^T y + R\frac{a_i^T a'_i}{\|a'_i\|} \leq b_i, \\ & i \in \{1, \cdots, p-1, p+1, \cdots, m\} \\ & a'_i = a_i - (a_i^T a_p)a_p \\ & a_p^T y = b_p \\ & R \geq 0 \end{aligned} \tag{2}$$

whereby R is the radius of the circle, y the center, a'_i is the orthogonal projection onto the hyperplane of a_p.

References

[Boyd and Vandenberghe(2004)] Stephen Boyd and Lieven Vandenberghe. *Convex Optimization*. Cambridge university press, 2004.