Geometrical visualization of the projection of a point onto a hyperplane

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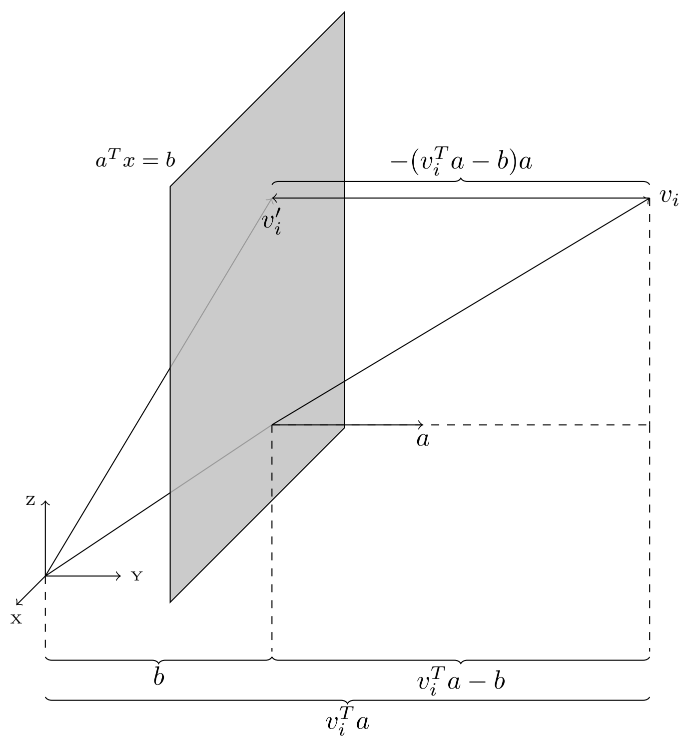

Figure 1: Geometrical visualization of the orthogonal projection of a point v_i onto a hyperplane H (defined by $a^T x = b$) given by $v_i' = v_i - (v_i^T a - b)a$.

In this document, I derive the projection of a point v in $\mathbb{R}^3$ onto a hyperplane H. I consider different cases, including an orthogonal projection, a projection along a given line L, and along a given line L parallel to the z-axis.

1 Orthogonal Projection

Given a point $v_i \in \mathbb{R}^N$, we like to find the orthogonal projection onto a hyperplane $H = \{x \in \mathbb{R}^N | a^T x = b, \|a\| = 1\}$. This can be obtained in the following way (see Fig. **??**): First we observe that the projection has to be along the direction of the hyperplane normal from v_i, i.e. this is in the form of $v_i' = v_i - \lambda a$, whereby λ is the distance of v_i towards the hyperplane. Since the distance of the hyperplane to the origin along a is given by b, and the distance of v_i to the origin along a is given by $v_i^T a$, we obtain the distance $\lambda = v_i^T a - b$.

2 Projection along line L

Let $L = \{x \in \mathbb{R}^N | x = c + \lambda(d - c)\}$ be a line through points $c, d \in \mathbb{R}^N$ and let $H = \{x \in \mathbb{R}^N | a^T x = b\}$ be a hyperplane. We are interested in finding the intersection of L with H, because this is the point where v is projected to. The intersection is found at the point where both equations become equivalent. Thus, we can derive this as

$$\begin{aligned} a^T(c + \lambda(d - c)) &= b \\ \lambda a^T(d - c) &= b - a^T c \\ \lambda &= \frac{b - a^T c}{a^T(d - c)} \quad | a^T(d - c) \neq 0 \end{aligned} \tag{1}$$

From this derivation, we can compute the point where L and H intersect as

$$x_{LH} = c + \frac{b - a^T c}{a^T(d - c)}(d - c) \tag{2}$$

for the case that $a^T(d-c) \neq 0$. If $a^T(d-c) = 0$, then the line is perpendicular to the plane normal, and thus parallel to the plane, i.e. either it does not intersect the plane, or it lies inside the hyperplane.

3 Projection of a point $v \in \mathbb{R}^3$ along line parallel to z-axis

Let $v \in \mathbb{R}^3$ be given. We are interested in finding the projection of v along a line L—through v and parallel to the z-axis—onto a hyperplane H. The expressions for L and H are given as

$$\begin{aligned} L &= \{x \in \mathbb{R}^3 | x = v - \lambda(0, 0, 1)^T\} \\ H &= \{x \in \mathbb{R}^N | a^T x = b\} \end{aligned}$$

Assume further that $a^T(0,0,1)^T \neq 0$, meaning the hyperplane is not parallel to the line. If the hyperplane is parallel, then either v lies on H or it cannot be projected. Then we can derive $\lambda = \frac{b - a^T v}{a^T(0,0,1)^T}$. Thus the projection of v onto H along L is given by

$$v' = v - \frac{b - a^T v}{a^T(0,0,1)^T}(0,0,1)^T$$